A Very Rough Introduction to Linear Logic

John Wickerson, Imperial College London

Multicore Group Seminar

January 7, 2014
• Introduced by Jean-Yves Girard in 1987
- Linear Logic
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- Classical logic: is my formula \textit{true}?
Linear Logic

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- Classical logic: is my formula *true*?
- Intuitionistic logic: is my formula *provable*?
Linear Logic

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- Classical logic: is my formula \textit{true}?
- Intuitionistic logic: is my formula \textit{provable}?
- Linear logic is a bit different
Vending machines

Let:

\[ D := \text{“One dollar”} \]
\[ M := \text{“A pack of Marlboros”} \]
\[ C := \text{“A pack of Camels”} \]
Vending machines

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Some axioms for a vending machine:

\[ D \Rightarrow M \]

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Vending machines

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Some axioms for a vending machine:

\[
D \Rightarrow M \\
D \Rightarrow C
\]

In classical or intuitionistic logic, we can deduce:

\[
D \Rightarrow (M \land C)
\]
Two conjunctions:

- **Multiplicative conjunction** \((A \otimes B)\):
  
  I can have both \(A\) and \(B\) at the same time

- **Additive conjunction** \((A \& B)\):
  
  I can have \(A\) or \(B\) (not both), and I choose which
Linear logic operators

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Two disjunctions:

- **Multiplicative disjunction** ($A \nabla B$):
  
  [we’ll come back to this one]

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  I can have \(A\) or \(B\) (not both), but I don’t choose which

De Morgan rules:

\[
(A \otimes B) ^\perp \equiv A ^\perp \nabla B ^\perp \\
(A & B) ^\perp \equiv A ^\perp \oplus B ^\perp
\]
More vending machines

Let $R$ be a vending machine that accepts rubles and dispenses packs of rolos, and let $S$ be a vending machine that accepts shillings and dispenses packs of softmints.
More vending machines

Let $R$ be a vending machine that accepts rubles and dispenses packs of rollos, and let $S$ be a vending machine that accepts shillings and dispenses packs of softmints. Let $P_1$, $P_2$, $P_3$ and $P_4$ be four purchase orders.
More vending machines

Let $R$ be a vending machine that accepts rubles and dispenses packs of rolos, and let $S$ be a vending machine that accepts shillings and dispenses packs of softmints. Let $P_1$, $P_2$, $P_3$ and $P_4$ be four purchase orders.

- $P_1 \rightarrow (R \otimes S)$: If Ally hands in purchase order $P_1$, the college will install both machines side-by-side.

- $P_2 \rightarrow (R \& S)$: If Ally hands in purchase order $P_2$, the college will install one machine of Ally's choice.

- $P_3 \rightarrow (R \oplus S)$: If Ally hands in purchase order $P_3$, the college will install one machine of their choice.

- $P_4 \rightarrow (R \` S)$: If Ally hands in purchase order $P_4$, the college will install both machines side-by-side, and pre-load one of them with a coin. If a customer puts a coin in the other slot, one of the machines will consume its coin and dispense its confectionery!
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Exponentials

- !A: Produces an unlimited number of A’s
- ?A: Consumes an unlimited number of A’s
- Connected via linear negation: (\(\neg A\)) = !(A\(\neg\))
## Jeroen’s Burger Shack

<table>
<thead>
<tr>
<th>Set menu – $5 per person</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamburger</td>
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## Jeroen’s Burger Shack

**Set menu – $5 per person**

- Hamburger
- Fries or Wedges
- Unlimited Pepsi
- Ice-cream or sorbet (subject to availability)

\[ D \otimes D \otimes D \otimes D \otimes D \otimes D \]

\[ \vdash \]

\[ H \otimes (F \& W) \otimes !P \otimes (I \oplus S) \]
The sequent

\[ A_1, \ldots, A_m \vdash B_1, \ldots, B_n \]

means

\[ (A_1 \land \cdots \land A_m) \Rightarrow (B_1 \lor \cdots \lor B_n) \]
Gentzen’s LK vs. Linear Logic

\[ \frac{\text{Axiom}}{\Gamma, A, A \vdash \Delta} \]

\[ \frac{\Gamma \vdash \Delta, A}{\Gamma, A \vdash \Delta} \quad \text{CONT}_L \]

\[ \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \quad \text{CONT}_R \]

\[ \frac{\Gamma \vdash \Delta, A, A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \quad \text{Cut} \]

\[ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \text{W}_L \]

\[ \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash A, \Delta} \quad \text{W}_R \]

\[ \frac{\Gamma, A \vdash \Delta, \Gamma, B \vdash \Delta}{\Gamma, A \lor B \vdash \Delta} \quad \text{\lor}_L \]

\[ \frac{\Gamma \vdash A, \Delta, A \lor B, \Delta}{\Gamma \vdash A \lor B, \Delta} \quad \text{\lor}_{R1} \]

\[ \frac{\Gamma \vdash B, \Delta, A \lor B, \Delta}{\Gamma \vdash A \lor B, \Delta} \quad \text{\lor}_{R2} \]

\[ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \land B \vdash \Delta} \quad \text{\land}_{L1} \]

\[ \frac{\Gamma, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta} \quad \text{\land}_{L2} \]

\[ \frac{\Gamma \vdash A, A \land B, \Delta}{\Gamma \vdash A \land B, \Delta} \quad \text{\land}_R \]
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\[ \frac{}{A \vdash A} \text{Axiom} \]

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\[ \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{Cont}_R \]

\[ \frac{\Gamma \vdash \Delta, A}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{Cut} \]

\[ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{W}_L \]

\[ \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{W}_R \]

\[ \frac{\Gamma, A \vdash \Delta, B \vdash \Delta}{\Gamma, A \lor B \vdash \Delta} \text{\lor}_L \]

\[ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \lor B, \Delta} \text{\lor}_{R1} \]

\[ \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \lor B, \Delta} \text{\lor}_{R2} \]

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\[ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \land B, \Delta} \text{\land}_R \]

\[ \frac{\Gamma, A \vdash \Delta, B \vdash \Delta'}{\Gamma, \Gamma', A \lor B \vdash \Delta, \Delta'} \text{\lor'}_L \]

\[ \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \lor B, \Delta} \text{\lor'}_R \]

\[ \frac{\Gamma, A, B \vdash \Delta, \Delta'}{\Gamma, A \land B \vdash \Delta} \text{\land'}_L \]

\[ \frac{\Gamma \vdash A, \Delta, B \vdash \Delta'}{\Gamma, \Gamma' \vdash A \land B, \Delta, \Delta'} \text{\land'}_R \]
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\[ \Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta \]
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\[ \Gamma \vdash B, \Delta \quad \Gamma \vdash A \lor B, \Delta \quad \text{\( \lor_{R2} \)} \]

\[ \Gamma, A \vdash \Delta \]
\[ \Gamma, A \land B \vdash \Delta \quad \text{\( \land_L \)} \]
\[ \Gamma, B \vdash \Delta \]
\[ \Gamma, A \land B \vdash \Delta \quad \text{\( \land_{L2} \)} \]
\[ \Gamma \vdash A, \Delta \]
\[ \Gamma \vdash A \land B, \Delta \quad \text{\( \land_R \)} \]

\[ \Gamma, A, B \vdash \Delta \]
\[ \Gamma, A \land B \vdash \Delta \quad \text{\( \land'_L \)} \]
\[ \Gamma', B \vdash \Delta' \]
\[ \Gamma, \Gamma', A \lor B \vdash \Delta, \Delta' \quad \text{\( \lor'_L \)} \]
\[ \Gamma \vdash A, \Delta \]
\[ \Gamma \vdash B, \Delta' \quad \text{\( \lor'_R \)} \]
\[ \Gamma \vdash A, \Delta \]
\[ \Gamma, \Gamma' \vdash A \land B, \Delta, \Delta' \quad \text{\( \land'_R \)} \]
Gentzen’s LK vs. Linear Logic

\[ \frac{A \vdash A}{A \vdash A} \quad \text{Axiom} \]

\[ \frac{\Gamma, \Delta \vdash A, A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \quad \text{Cut} \]

\[ \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \quad \oplus_L \]

\[ \frac{\Gamma \vdash A, \Delta}{\Gamma, A \oplus B, \Delta} \quad \oplus_{R1} \]

\[ \frac{\Gamma \vdash B, \Delta}{\Gamma, A \oplus B, \Delta} \quad \oplus_{R2} \]

\[ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \quad \&_{L1} \]

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\[ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \& B, \Delta} \quad \&_R \]

\[ \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \otimes B \vdash \Delta, \Delta'} \quad \otimes_L \]

\[ \frac{\Gamma \vdash A, \Delta}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \quad \otimes_R \]

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Gentzen’s LK vs. Linear Logic

\[ \begin{array}{cccc}
\Gamma, A \vdash A & \text{Axiom} & \Gamma, !A \vdash \Delta & \text{Cont}_L \\
\hline
\dashv & & \hline
\end{array} \]

\[ \begin{array}{cccc}
\Gamma \vdash \Delta, A & A, \Gamma' \vdash \Delta' & \text{Cut} & \Gamma \vdash \Delta & \text{W}_L \\
\hline
\Gamma, \Gamma' \vdash \Delta, \Delta' & \text{Cont}_R & \Gamma \vdash !A \vdash \Delta & \text{W}_R \\
\end{array} \]

\[ \begin{array}{cccc}
\Gamma, A \vdash \Delta & \Gamma, B \vdash \Delta & \oplus_L & \Gamma \vdash A, \Delta \vdash \Delta & \oplus_{R1} \\
\hline
\Gamma, A \oplus B \vdash \Delta & \Gamma \vdash A \oplus B, \Delta & \oplus_{R2} & \end{array} \]

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\Gamma, A \vdash \Delta & \&_{L1} & \Gamma, B \vdash \Delta & \&_{L2} \\
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\Gamma, A, B \vdash \Delta & \Delta_L & \Gamma \vdash A \& B, \Delta & \Delta_R \\
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\hline
\Gamma, A \& B \vdash \Delta & \end{array} \]
Application: Petri nets

\[ \begin{array}{c}
\begin{array}{c}
A \\
\bullet \\
\end{array}
\end{array}
\quad \quad
\begin{array}{c}
\begin{array}{c}
B \\
\bullet \\
\end{array}
\end{array}
\quad \quad
\begin{array}{c}
\begin{array}{c}
C \\
\bullet \\
\end{array}
\end{array}
\quad \quad
\begin{array}{c}
\begin{array}{c}
D \\
\bullet \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
A \\
\bullet \\
\end{array}
\end{array}
\quad \quad
\begin{array}{c}
\begin{array}{c}
B \\
\bullet \\
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\end{array}
\quad \quad
\begin{array}{c}
\begin{array}{c}
C \\
\bullet \\
\end{array}
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\quad \quad
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\begin{array}{c}
D \\
\bullet \\
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\end{array}
\end{array} \]

Other applications include: finding optimisations in Haskell, logic programming, quantum physics, linguistics, . . .
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Application: Petri nets

\[\text{!}((A \otimes C) \multimap B) \otimes \text{!}((B \otimes D \otimes D) \multimap (C \otimes D)) \otimes A \otimes C \otimes D \otimes D\]
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Linear Logic
Application: Petri nets

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Linear Logic
Application: Petri nets

![Petri net diagram]

\[
\begin{array}{l}
!( (A \otimes C) \multimap B ) \otimes \\
!( (B \otimes D \otimes D) \multimap (C \otimes D) ) \otimes \\
(A \otimes C \otimes D \otimes D)
\end{array}
\multimap (C \otimes D)
\]

- Other applications include: finding optimisations in Haskell, logic programming, quantum physics, linguistics, …
Connection to separation logic

- Linear logic models the *number of times* a resource is used.
- To reason about computer memory, we want to control how resources are *shared*.
- The logic of *bunched implications* (BI) does this. (Separation logic is based on BI.)
The idea of BI

- Sequents in classical/linear logic have this form:

\[ \Gamma \vdash \Gamma \]

where \( \Gamma ::= A | \Gamma, \Gamma \)
The idea of BI

- Sequents in classical/linear logic have this form:
  \[ \Gamma \vdash \Gamma \]
  where \( \Gamma ::= A \mid \Gamma, \Gamma \)
- Sequents in intuitionistic logic have this form:
  \[ \Gamma \vdash A \]
The idea of BI

- Sequents in classical/linear logic have this form:

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  where \( \Gamma ::= A \mid \Gamma, \Gamma \)

- Sequents in intuitionistic logic have this form:

  \[ \Gamma \vdash A \]

- Sequents in BI have this form:

  \[ \mathcal{B} \vdash A \]

  where \( \mathcal{B} ::= A \mid \mathcal{B}, \mathcal{B} \mid \mathcal{B}; \mathcal{B} \)
Some proof rules for BI

- Weakening and Contraction are allowed for ‘;’ but not for ‘,’

\[
B \vdash A \\
\frac{B \vdash A}{B ; B' \vdash A} \text{ W} \\
\frac{B ; B \vdash A}{B \vdash A} \text{ CONT}
\]
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\[
\frac{B \vdash A}{B; B' \vdash A} \text{ W} \quad \frac{B; B \vdash A}{B \vdash A} \text{ CONT}
\]

- The ‘;’ corresponds to ‘∧’ (ordinary conjunction)

\[
\frac{B \vdash A \quad B' \vdash A'}{B; B' \vdash A \land A'} \text{ ^I}
\]

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Some proof rules for BI

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\]

- The ‘;’ corresponds to ‘∧’ (ordinary conjunction)

\[
\frac{\mathcal{B} \vdash A \quad \mathcal{B}' \vdash A'}{\mathcal{B}; \mathcal{B}' \vdash A \land A'} \land I
\]

- The ‘,’ corresponds to ‘∗’ (separating conjunction)

\[
\frac{\mathcal{B} \vdash A \quad \mathcal{B}' \vdash A'}{\mathcal{B}, \mathcal{B}' \vdash A \ast A'} \ast I
\]
Conclusion

Linear logic . . .

- is ‘a logic behind logics’
Linear logic ...

- is ‘a logic behind logics’
- models production and consumption of resources (but not sharing)
Linear logic . . .

- is ‘a logic behind logics’
- models production and consumption of resources (but not sharing)
- has two types of conjunction, and two types of disjunction
Linear logic . . .

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- models production and consumption of resources (but not sharing)
- has two types of conjunction, and two types of disjunction
- has exponentials, which relax the strict rules on resource usage
Conclusion

Linear logic . . .

- is ‘a logic behind logics’
- models production and consumption of resources (but not sharing)
- has two types of conjunction, and two types of disjunction
- has exponentials, which relax the strict rules on resource usage
- is obtained by removing **Weakening** and **Contraction** from classical logic
Linear logic . . .

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- models production and consumption of resources (but not sharing)
- has two types of conjunction, and two types of disjunction
- has exponentials, which relax the strict rules on resource usage
- is obtained by removing Weakening and Contraction from classical logic
- has various applications, e.g. modelling Petri nets
Bibliography

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Linear logic.
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The logic of bunched implications.

Introduction to proof theory.