Nominal Unification: an introduction

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Based on the 2004 article *Nominal Unification*
by Christian Urban, Andrew Pitts and Jamie Gabbay

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Warmup: do these terms unify?

$$P \wedge Q \quad \approx? \quad (\mathit{true} \vee \mathit{false}) \wedge (\mathit{false} \Rightarrow \mathit{true})$$

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$$\begin{aligned} P \wedge Q &\approx? \quad (\mathit{true} \vee \mathit{false}) \wedge (\mathit{false} \Rightarrow \mathit{true}) \\ \texttt{int} \to T_1 &\approx? \quad T_2 \to (T_3 \times T_4) \end{aligned}$$

Warmup: do these terms unify?

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\begin{aligned}
P \wedge Q &\approx ? \quad (\mathit{true} \vee \mathit{false}) \wedge (\mathit{false} \Rightarrow \mathit{true}) \\
\texttt{int} \to T_1 &\approx ? \quad T_2 \to (T_3 \times T_4) \\
j(a, g(X, f(Y)), h(Y)) &\approx ? \quad j(X, g(Y, Z), h(b))
\end{aligned}
$$

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P \wedge Q \quad &\approx? \quad (\mathit{true} \vee \mathit{false}) \wedge (\mathit{false} \Rightarrow \mathit{true}) \\
\mathtt{int} \rightarrow T_1 \quad &\approx? \quad T_2 \rightarrow (T_3 \times T_4) \\
j(a, g(X, f(Y)), h(Y)) \quad &\approx? \quad j(X, g(Y, Z), h(b)) \\
f(Y, h(a, b), Y) \quad &\approx? \quad f(X, h(V, Z), g(X))
\end{aligned}
$$

Unification

- Input: two terms, t_1 and t_2

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Unification

- Input: two terms, t_1 and t_2
- Output: a substitution σ where $\sigma(t_1) = \sigma(t_2)$
- Moreover, σ should be the 'most general' solution; that is, for any other solution σ', there exists a substitution σ'' that makes $\sigma'(t) = \sigma''(\sigma(t))$ for all t

Namey terms

$$\forall x.\, x < 5 \quad = \quad \forall y.\, y < 5$$

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Definition (Syntax of λ-terms)

$$lam ::= \lambda x.\, lam \mid lam\ lam \mid x$$

Namey terms

$$\begin{aligned} \forall x.\, x < 5 &\quad=\quad \forall y.\, y < 5 \\ \{x \mid x < z\} &\quad\neq\quad \{y \mid y < w\} \\ \lambda x.\, \lambda y.\, y + 2 &\quad=\quad \lambda x.\, \lambda x.\, x + 2 \\ (\lambda x.\, (\lambda x.\, x)(\lambda x.\, x)x)x &\quad=\quad (\lambda y.\, (\lambda z.\, z)(\lambda w.\, w)y)x \end{aligned}$$

Definition (Syntax of λ-terms)

$$lam ::= \lambda x.\, lam \mid lam\ lam \mid x$$

Do these namey terms unify?

$$\lambda x.\,\lambda y.\,Z_1\,y \quad \approx? \quad \lambda y.\,\lambda x.\,x\,Z_1$$

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Do these namey terms unify?

$$\begin{aligned}
\lambda x.\, \lambda y.\, Z_1\, y &\approx? \quad \lambda y.\, \lambda x.\, x\, Z_1 \\
\lambda x.\, \lambda y.\, Z_2\, y &\approx? \quad \lambda y.\, \lambda x.\, x\, Z_3 \\
\lambda x.\, \lambda y.\, y\, Z_4 &\approx? \quad \lambda y.\, \lambda x.\, x\, Z_5
\end{aligned}$$

Do these namey terms unify?

$$\begin{aligned}
\lambda x.\,\lambda y.\,Z_1\,y &\approx? \quad \lambda y.\,\lambda x.\,x\,Z_1 \\
\lambda x.\,\lambda y.\,Z_2\,y &\approx? \quad \lambda y.\,\lambda x.\,x\,Z_3 \\
\lambda x.\,\lambda y.\,y\,Z_4 &\approx? \quad \lambda y.\,\lambda x.\,x\,Z_5 \\
\lambda x.\,\lambda y.\,y\,Z_6 &\approx? \quad \lambda x.\,\lambda x.\,x\,Z_7
\end{aligned}$$

This talk

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I will describe an algorithm that:

- takes a pair of λ-terms, M_1 and M_2, and
- returns the most general unifier of M_1 and M_2 (if it exists)

Nominal unification: problem specification

Input: a finite set P of

- *equational problems* (e.g. $M_1 \approx ? \, M_2$) and
- *freshness problems* (e.g. $x \,\#? \, M$)

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Input: a finite set P of

- *equational problems* (e.g. $M_1 \approx ?\ M_2$) and
- *freshness problems* (e.g. $x \mathrel{\#?} M$)

Output: a pair (∇, σ), where ∇ is a finite set of *freshness assumptions* and σ is a substitution, that satisfies:

- $\nabla \vdash x \mathrel{\#} \sigma(M)$ for each $(x \mathrel{\#?} M)$ in P,
- $\nabla \vdash \sigma(M_1) = \sigma(M_2)$ for each $(M_1 \approx ?\ M_2)$ in P

Nominal unification: problem specification

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- *equational problems* (e.g. $M_1 \approx ?\ M_2$) and
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Output: a pair (∇, σ), where ∇ is a finite set of *freshness assumptions* and σ is a substitution, that satisfies:

- $\nabla \vdash x \,\#\, \sigma(M)$ for each $(x \,\#?\ M)$ in P,
- $\nabla \vdash \sigma(M_1) = \sigma(M_2)$ for each $(M_1 \approx ?\ M_2)$ in P, and
- for any other solution (∇', σ') there exists a substitution σ'' that makes
  - $\nabla' \vdash x \,\#\, \sigma''(Z)$ for each $(x \,\#\, Z)$ in ∇' and
  - $\nabla' \vdash \sigma'(M) = \sigma''(\sigma(M))$ for all M.

Nominal unification: algorithm outline

- Repeatedly transforms the set of problems
- First applies $\stackrel{\sigma}{\Longrightarrow}$ until no equational problems remain
- Then applies $\stackrel{\nabla}{\Longrightarrow}$ until no problems remain
- Might fail in either stage
- Guaranteed to terminate
- Successful transformation sequence on input P is of the form

$$P \stackrel{\sigma_1}{\Longrightarrow} \cdots \stackrel{\sigma_n}{\Longrightarrow} P' \stackrel{\nabla_1}{\Longrightarrow} \cdots \stackrel{\nabla_m}{\Longrightarrow} \emptyset$$

- Returns solution $(\nabla_1 \cup \cdots \cup \nabla_m, \sigma_n \circ \cdots \circ \sigma_1)$

Permutations

A permutation π is a list of pairs of variables:

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Definition (Applying a permutation to a λ-term)

$$\begin{array}{rcl}
\pi\,(\lambda x.\, M) & = & \lambda(\pi\, x).\,(\pi\, M) \\
\pi\,(M_1\, M_2) & = & (\pi\, M_1)\,(\pi\, M_2) \\
[\,]\, x & = & x \\
((y \leftrightarrow z) :: \pi)\, x & = & \begin{cases} z & \text{if } \pi\, x = y \\ y & \text{if } \pi\, x = z \\ \pi\, x & \text{otherwise} \end{cases} \\
\pi\,(\pi'\, Z) & = & (\pi \mathbin{@} \pi')\, Z
\end{array}$$

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A permutation π is a list of pairs of variables:

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$$\begin{array}{rcl} \pi\,(\lambda x.\, M) & = & \lambda(\pi\, x).\,(\pi\, M) \\ \pi\,(M_1\, M_2) & = & (\pi\, M_1)\,(\pi\, M_2) \\ [\,]\, x & = & x \\ ((y \leftrightarrow z) :: \pi)\, x & = & \begin{cases} z & \text{if } \pi\, x = y \\ y & \text{if } \pi\, x = z \\ \pi\, x & \text{otherwise} \end{cases} \\ \pi\,(\pi'\, Z) & = & (\pi \mathbin{@} \pi')\, Z \end{array}$$

Definition (Syntax of λ-terms with ‘suspensions’)

$$\mathit{lam} ::= \lambda x.\, \mathit{lam} \mid \mathit{lam}\ \mathit{lam} \mid x \mid \pi\, Z$$

Phase 1 transformations

$\approx$?-fun-1: $\{\lambda x.\, M \approx ?\ \lambda x.\, M'\} \uplus P \overset{\epsilon}{\Longrightarrow} \{M \approx ?\ M'\} \cup P$

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$$\approx?\text{-fun-1: } \{\lambda x.\, M \approx ?\ \lambda x.\, M'\} \uplus P \overset{\epsilon}{\Longrightarrow} \{M \approx ?\ M'\} \cup P$$

$$\approx?\text{-fun-2: } \{\lambda x.\, M \approx ?\ \lambda y.\, M'\} \uplus P \overset{\epsilon}{\Longrightarrow} \{M \approx ?\ (x \leftrightarrow y)\, M', x \,\#?\ M'\} \cup P$$

$$\text{(where } x \neq y\text{)}$$

Phase 1 transformations

$\approx$?-fun-1: $\{\lambda x.\, M \approx ?\ \lambda x.\, M'\} \uplus P \overset{\epsilon}{\Longrightarrow} \{M \approx ?\ M'\} \cup P$

$\approx$?-fun-2: $\{\lambda x.\, M \approx ?\ \lambda y.\, M'\} \uplus P \overset{\epsilon}{\Longrightarrow} \{M \approx ?\ (x \leftrightarrow y)\, M', x \,\#?\, M'\} \cup P$

(where $x \neq y$)

$\approx$?-app: $\{M_1\, M_2 \approx ?\ M_1'\, M_2'\} \uplus P \overset{\epsilon}{\Longrightarrow} \{M_1 \approx ?\ M_1', M_2 \approx ?\ M_2'\} \cup P$

Phase 1 transformations

$$
\begin{array}{lrl}
\approx?\text{-fun-1:} & \{\lambda x.\, M \approx?\ \lambda x.\, M'\} \uplus P & \overset{\epsilon}{\Longrightarrow} \{M \approx?\ M'\} \cup P \\
\approx?\text{-fun-2:} & \{\lambda x.\, M \approx?\ \lambda y.\, M'\} \uplus P & \overset{\epsilon}{\Longrightarrow} \{M \approx?\ (x \leftrightarrow y)\, M', x \,\#?\ M'\} \cup P \\
 & & \hfill (\text{where } x \neq y) \\
\approx?\text{-app:} & \{M_1\, M_2 \approx?\ M_1'\, M_2'\} \uplus P & \overset{\epsilon}{\Longrightarrow} \{M_1 \approx?\ M_1', M_2 \approx?\ M_2'\} \cup P \\
\approx?\text{-var:} & \{x \approx?\ x\} \uplus P & \overset{\epsilon}{\Longrightarrow} P
\end{array}
$$

Phase 1 transformations

$$\approx?\text{-fun-1: } \{\lambda x.\, M \approx?\ \lambda x.\, M'\} \uplus P \stackrel{\epsilon}{\Longrightarrow} \{M \approx?\ M'\} \cup P$$

$$\approx?\text{-fun-2: } \{\lambda x.\, M \approx?\ \lambda y.\, M'\} \uplus P \stackrel{\epsilon}{\Longrightarrow} \{M \approx?\ (x \leftrightarrow y)\, M', x \ \#?\ M'\} \cup P \quad (\text{where } x \neq y)$$

$$\approx?\text{-app: } \{M_1\, M_2 \approx?\ M_1'\, M_2'\} \uplus P \stackrel{\epsilon}{\Longrightarrow} \{M_1 \approx?\ M_1', M_2 \approx?\ M_2'\} \cup P$$

$$\approx?\text{-var: } \{x \approx?\ x\} \uplus P \stackrel{\epsilon}{\Longrightarrow} P$$

$$\approx?\text{-susp-1: } \{\pi\, Z \approx?\ \pi'\, Z\} \uplus P \stackrel{\epsilon}{\Longrightarrow} \{(x \ \#?\ Z) \mid \pi\, x \neq \pi'\, x\} \cup P$$

Phase 1 transformations

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\begin{array}{lrl}
\approx?\text{-fun-1:} & \{\lambda x.\, M \approx?\ \lambda x.\, M'\} \uplus P & \overset{\epsilon}{\Longrightarrow} \{M \approx?\ M'\} \cup P \\
\approx?\text{-fun-2:} & \{\lambda x.\, M \approx?\ \lambda y.\, M'\} \uplus P & \overset{\epsilon}{\Longrightarrow} \{M \approx?\ (x \leftrightarrow y)\, M', x \#?\ M'\} \cup P \\
 & & \hfill (\text{where } x \neq y) \\
\approx?\text{-app:} & \{M_1\, M_2 \approx?\ M_1'\, M_2'\} \uplus P & \overset{\epsilon}{\Longrightarrow} \{M_1 \approx?\ M_1', M_2 \approx?\ M_2'\} \cup P \\
\approx?\text{-var:} & \{x \approx?\ x\} \uplus P & \overset{\epsilon}{\Longrightarrow} P \\
\approx?\text{-susp-1:} & \{\pi\, Z \approx?\ \pi'\, Z\} \uplus P & \overset{\epsilon}{\Longrightarrow} \{(x \#?\ Z) \mid \pi\, x \neq \pi'\, x\} \cup P \\
\approx?\text{-susp-2:} & \{M \approx?\ \pi\, Z\} \uplus P & \overset{\sigma}{\Longrightarrow} \sigma(P) \\
 & \multicolumn{2}{r}{(\text{where } M \text{ does not mention } Z, \text{ and } \sigma = [Z := \pi^{-1}\, M])}
\end{array}
$$

Phase 2 transformations

#?-fun-1: $\{x \,\#?\, \lambda x.\, M\} \uplus P \xrightarrow{\emptyset} P$

Phase 2 transformations

$$\text{\#?-fun-1:} \quad \{x \mathrel{\#?} \lambda x.\, M\} \uplus P \stackrel{\emptyset}{\Longrightarrow} P$$

$$\text{\#?-fun-2:} \quad \{x \mathrel{\#?} \lambda y.\, M\} \uplus P \stackrel{\emptyset}{\Longrightarrow} \{x \mathrel{\#?} M\} \cup P \quad (\text{where } x \neq y)$$

Phase 2 transformations

$$
\begin{array}{llcl}
\#?\text{-fun-1:} & \{x \mathrel{\#?} \lambda x.\, M\} \uplus P & \overset{\emptyset}{\Longrightarrow} & P \\
\#?\text{-fun-2:} & \{x \mathrel{\#?} \lambda y.\, M\} \uplus P & \overset{\emptyset}{\Longrightarrow} & \{x \mathrel{\#?} M\} \cup P \quad (\text{where } x \neq y) \\
\#?\text{-app:} & \{x \mathrel{\#?} M_1\, M_2\} \uplus P & \overset{\emptyset}{\Longrightarrow} & \{x \mathrel{\#?} M_1, x \mathrel{\#?} M_2\} \cup P
\end{array}
$$

Phase 2 transformations

$$
\begin{array}{llcll}
\#?\text{-fun-1:} & \{x \;\#?\; \lambda x.\, M\} \uplus P & \overset{\emptyset}{\Longrightarrow} & P & \\
\#?\text{-fun-2:} & \{x \;\#?\; \lambda y.\, M\} \uplus P & \overset{\emptyset}{\Longrightarrow} & \{x \;\#?\; M\} \cup P & (\text{where } x \neq y) \\
\#?\text{-app:} & \{x \;\#?\; M_1\, M_2\} \uplus P & \overset{\emptyset}{\Longrightarrow} & \{x \;\#?\; M_1, x \;\#?\; M_2\} \cup P & \\
\#?\text{-var:} & \{x \;\#?\; y\} \uplus P & \overset{\emptyset}{\Longrightarrow} & P & (\text{where } x \neq y)
\end{array}
$$

Phase 2 transformations

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\begin{array}{llcll}
\#?\text{-fun-1:} & \{x \;\#?\; \lambda x.\, M\} \uplus P & \overset{\emptyset}{\Longrightarrow} & P & \\
\#?\text{-fun-2:} & \{x \;\#?\; \lambda y.\, M\} \uplus P & \overset{\emptyset}{\Longrightarrow} & \{x \;\#?\; M\} \cup P & (\text{where } x \neq y) \\
\#?\text{-app:} & \{x \;\#?\; M_1\, M_2\} \uplus P & \overset{\emptyset}{\Longrightarrow} & \{x \;\#?\; M_1, x \;\#?\; M_2\} \cup P & \\
\#?\text{-var:} & \{x \;\#?\; y\} \uplus P & \overset{\emptyset}{\Longrightarrow} & P & (\text{where } x \neq y) \\
\#?\text{-susp:} & \{x \;\#?\; \pi\, Z\} \uplus P & \overset{\nabla}{\Longrightarrow} & P & (\text{where } \nabla = \{\pi^{-1} x \;\#\; Z\})
\end{array}
$$

Worked example 1

$$\{\lambda x.\, \lambda y.\, Z_1\, y \approx ?\ \lambda y.\, \lambda x.\, x\, Z_1\}$$

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$$\overset{\epsilon}{\Longrightarrow} \quad \{\lambda y.\, Z_1\, y \approx?\ (x \leftrightarrow y)\, (\lambda x.\, x\, Z_1),\, x \ \#?\ \lambda x.\, x\, Z_1\} \qquad (\approx?\text{-fun-2})$$

Worked example 1

$$
\begin{aligned}
& \{\lambda x.\, \lambda y.\, Z_1\, y \approx?\ \lambda y.\, \lambda x.\, x\, Z_1\} \\
\stackrel{\epsilon}{\Longrightarrow}\ & \{\lambda y.\, Z_1\, y \approx?\ (x \leftrightarrow y)\, (\lambda x.\, x\, Z_1), x \#?\ \lambda x.\, x\, Z_1\} \quad (\approx?\text{-fun-2}) \\
=\ & \{\lambda y.\, Z_1\, y \approx?\ \lambda y.\, y\, ((x \leftrightarrow y)\, Z_1), x \#?\ \lambda x.\, x\, Z_1\}
\end{aligned}
$$

$$
\begin{array}{rll}
 & \{\lambda x.\,\lambda y.\,Z_1\,y \approx ?\ \lambda y.\,\lambda x.\,x\,Z_1\} & \\
\overset{\epsilon}{\Longrightarrow} & \{\lambda y.\,Z_1\,y \approx ?\ (x \leftrightarrow y)\,(\lambda x.\,x\,Z_1), x \ \#?\ \lambda x.\,x\,Z_1\} & (\approx?\text{-fun-2}) \\
= & \{\lambda y.\,Z_1\,y \approx ?\ \lambda y.\,y\,((x \leftrightarrow y)\,Z_1), x \ \#?\ \lambda x.\,x\,Z_1\} & \\
\overset{\epsilon}{\Longrightarrow} & \{Z_1\,y \approx ?\ y\,((x \leftrightarrow y)\,Z_1), x \ \#?\ \lambda x.\,x\,Z_1\} & (\approx?\text{-fun-1})
\end{array}
$$

Worked example 1

$$
\begin{array}{rll}
 & \{\lambda x.\, \lambda y.\, Z_1\, y \approx ?\ \lambda y.\, \lambda x.\, x\, Z_1\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{\lambda y.\, Z_1\, y \approx ?\ (x \leftrightarrow y)\, (\lambda x.\, x\, Z_1), x \#?\ \lambda x.\, x\, Z_1\} & (\approx ?\text{-fun-2}) \\
= & \{\lambda y.\, Z_1\, y \approx ?\ \lambda y.\, y\, ((x \leftrightarrow y)\, Z_1), x \#?\ \lambda x.\, x\, Z_1\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{Z_1\, y \approx ?\ y\, ((x \leftrightarrow y)\, Z_1), x \#?\ \lambda x.\, x\, Z_1\} & (\approx ?\text{-fun-1}) \\
\stackrel{\epsilon}{\Longrightarrow} & \{Z_1 \approx ?\ y, y \approx ?\ (x \leftrightarrow y)\, Z_1, x \#?\ \lambda x.\, x\, Z_1\} & (\approx ?\text{-app})
\end{array}
$$

Worked example 1

$$
\begin{array}{rll}
& \{\lambda x.\, \lambda y.\, Z_1\, y \approx ?\ \lambda y.\, \lambda x.\, x\, Z_1\} & \\
\overset{\epsilon}{\Longrightarrow} & \{\lambda y.\, Z_1\, y \approx ?\ (x \leftrightarrow y)\, (\lambda x.\, x\, Z_1), x \ \#?\ \lambda x.\, x\, Z_1\} & (\approx ?\text{-fun-2}) \\
= & \{\lambda y.\, Z_1\, y \approx ?\ \lambda y.\, y\, ((x \leftrightarrow y)\, Z_1), x \ \#?\ \lambda x.\, x\, Z_1\} & \\
\overset{\epsilon}{\Longrightarrow} & \{Z_1\, y \approx ?\ y\, ((x \leftrightarrow y)\, Z_1), x \ \#?\ \lambda x.\, x\, Z_1\} & (\approx ?\text{-fun-1}) \\
\overset{\epsilon}{\Longrightarrow} & \{Z_1 \approx ?\ y, y \approx ?\ (x \leftrightarrow y)\, Z_1, x \ \#?\ \lambda x.\, x\, Z_1\} & (\approx ?\text{-app}) \\
\overset{Z_1 := y}{\Longrightarrow} & \{y \approx ?\ (x \leftrightarrow y)\, y, x \ \#?\ \lambda x.\, x\, y\} & (\approx ?\text{-susp-2})
\end{array}
$$

Worked example 1

$$
\begin{array}{rll}
& \{\lambda x.\,\lambda y.\,Z_1\,y \approx?\ \lambda y.\,\lambda x.\,x\,Z_1\} & \\
\overset{\epsilon}{\Longrightarrow} & \{\lambda y.\,Z_1\,y \approx?\ (x \leftrightarrow y)\,(\lambda x.\,x\,Z_1),\, x \ \#?\ \lambda x.\,x\,Z_1\} & (\approx?\text{-fun-2}) \\
= & \{\lambda y.\,Z_1\,y \approx?\ \lambda y.\,y\,((x \leftrightarrow y)\,Z_1),\, x \ \#?\ \lambda x.\,x\,Z_1\} & \\
\overset{\epsilon}{\Longrightarrow} & \{Z_1\,y \approx?\ y\,((x \leftrightarrow y)\,Z_1),\, x \ \#?\ \lambda x.\,x\,Z_1\} & (\approx?\text{-fun-1}) \\
\overset{\epsilon}{\Longrightarrow} & \{Z_1 \approx?\ y,\, y \approx?\ (x \leftrightarrow y)\,Z_1,\, x \ \#?\ \lambda x.\,x\,Z_1\} & (\approx?\text{-app}) \\
\overset{Z_1:=y}{\Longrightarrow} & \{y \approx?\ (x \leftrightarrow y)\,y,\, x \ \#?\ \lambda x.\,x\,y\} & (\approx?\text{-susp-2}) \\
= & \{y \approx?\ x,\, x \ \#?\ \lambda x.\,x\,y\} &
\end{array}
$$

Worked example 1

$$
\begin{array}{rll}
& \{\lambda x.\, \lambda y.\, Z_1\, y \approx ?\ \lambda y.\, \lambda x.\, x\, Z_1\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{\lambda y.\, Z_1\, y \approx ?\ (x \leftrightarrow y)\, (\lambda x.\, x\, Z_1), x \ \#?\ \lambda x.\, x\, Z_1\} & (\approx?\text{-fun-2}) \\
= & \{\lambda y.\, Z_1\, y \approx ?\ \lambda y.\, y\, ((x \leftrightarrow y)\, Z_1), x \ \#?\ \lambda x.\, x\, Z_1\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{Z_1\, y \approx ?\ y\, ((x \leftrightarrow y)\, Z_1), x \ \#?\ \lambda x.\, x\, Z_1\} & (\approx?\text{-fun-1}) \\
\stackrel{\epsilon}{\Longrightarrow} & \{Z_1 \approx ?\ y, y \approx ?\ (x \leftrightarrow y)\, Z_1, x \ \#?\ \lambda x.\, x\, Z_1\} & (\approx?\text{-app}) \\
\stackrel{Z_1:=y}{\Longrightarrow} & \{y \approx ?\ (x \leftrightarrow y)\, y, x \ \#?\ \lambda x.\, x\, y\} & (\approx?\text{-susp-2}) \\
= & \{y \approx ?\ x, x \ \#?\ \lambda x.\, x\, y\} &
\end{array}
$$

FAIL!

Worked example 2

$$\{\lambda x.\, \lambda y.\, y\, Z_6 \approx ?\ \lambda x.\, \lambda x.\, x\, Z_7\}$$

Worked example 2

$$\{\lambda x.\, \lambda y.\, y\, Z_6 \approx ?\ \lambda x.\, \lambda x.\, x\, Z_7\}$$

$$\stackrel{\epsilon}{\Longrightarrow} \{\lambda y.\, y\, Z_6 \approx ?\ \lambda x.\, x\, Z_7\} \qquad (\approx?\text{-abs-1})$$

Worked example 2

$$\{\lambda x.\, \lambda y.\, y\, Z_6 \approx ?\ \lambda x.\, \lambda x.\, x\, Z_7\}$$

$$\stackrel{\epsilon}{\Longrightarrow} \{\lambda y.\, y\, Z_6 \approx ?\ \lambda x.\, x\, Z_7\} \qquad (\approx?\text{-abs-1})$$

$$\stackrel{\epsilon}{\Longrightarrow} \{y\, Z_6 \approx ?\ (y \leftrightarrow x)\,(x\, Z_7),\, y\ \#?\ x\, Z_7\} \qquad (\approx?\text{-abs-2})$$

Worked example 2

$$
\begin{array}{rll}
& \{\lambda x.\,\lambda y.\,y\,Z_6 \approx ?\ \lambda x.\,\lambda x.\,x\,Z_7\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{\lambda y.\,y\,Z_6 \approx ?\ \lambda x.\,x\,Z_7\} & (\approx?\text{-abs-1}) \\
\stackrel{\epsilon}{\Longrightarrow} & \{y\,Z_6 \approx ?\ (y \leftrightarrow x)\,(x\,Z_7), y\ \#?\ x\,Z_7\} & (\approx?\text{-abs-2}) \\
= & \{y\,Z_6 \approx ?\ y\,((y \leftrightarrow x)\,Z_7), y\ \#?\ x\,Z_7\} &
\end{array}
$$

Worked example 2

$$
\begin{array}{rll}
& \{\lambda x.\,\lambda y.\,y\,Z_6 \approx ?\ \lambda x.\,\lambda x.\,x\,Z_7\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{\lambda y.\,y\,Z_6 \approx ?\ \lambda x.\,x\,Z_7\} & (\approx?\text{-abs-1}) \\
\stackrel{\epsilon}{\Longrightarrow} & \{y\,Z_6 \approx ?\ (y \leftrightarrow x)\,(x\,Z_7), y\ \#?\ x\,Z_7\} & (\approx?\text{-abs-2}) \\
= & \{y\,Z_6 \approx ?\ y\,((y \leftrightarrow x)\,Z_7), y\ \#?\ x\,Z_7\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{y \approx ?\ y, Z_6 \approx ?\ (y \leftrightarrow x)\,Z_7, y\ \#?\ x\,Z_7\} & (\approx?\text{-app})
\end{array}
$$

Worked example 2

$$
\begin{array}{rll}
 & \{\lambda x.\, \lambda y.\, y\, Z_6 \approx ?\ \lambda x.\, \lambda x.\, x\, Z_7\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{\lambda y.\, y\, Z_6 \approx ?\ \lambda x.\, x\, Z_7\} & (\approx ?\text{-abs-1}) \\
\stackrel{\epsilon}{\Longrightarrow} & \{y\, Z_6 \approx ?\ (y \leftrightarrow x)\,(x\, Z_7),\, y \;\#?\; x\, Z_7\} & (\approx ?\text{-abs-2}) \\
= & \{y\, Z_6 \approx ?\ y\,((y \leftrightarrow x)\, Z_7),\, y \;\#?\; x\, Z_7\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{y \approx ?\ y,\, Z_6 \approx ?\ (y \leftrightarrow x)\, Z_7,\, y \;\#?\; x\, Z_7\} & (\approx ?\text{-app}) \\
\stackrel{\epsilon}{\Longrightarrow} & \{Z_6 \approx ?\ (y \leftrightarrow x)\, Z_7,\, y \;\#?\; x\, Z_7\} & (\approx ?\text{-var})
\end{array}
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= & \{y\,Z_6 \approx ?\ y\,((y \leftrightarrow x)\,Z_7), y \# ?\ x\,Z_7\} & \\
\stackrel{\epsilon}{\Longrightarrow} & \{y \approx ?\ y, Z_6 \approx ?\ (y \leftrightarrow x)\,Z_7, y \# ?\ x\,Z_7\} & (\approx?\text{-app}) \\
\stackrel{\epsilon}{\Longrightarrow} & \{Z_6 \approx ?\ (y \leftrightarrow x)\,Z_7, y \# ?\ x\,Z_7\} & (\approx?\text{-var}) \\
\stackrel{Z_6 := (y \leftrightarrow x)\,Z_7}{\Longrightarrow} & \{y \# ?\ x\,Z_7\} & (\approx?\text{-susp-2})
\end{array}
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\overset{\emptyset}{\Longrightarrow} & \{y \#?\ x, y \#?\ Z_7\} & (\#?\text{-app})
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\overset{\epsilon}{\Longrightarrow} & \{y \approx?\ y, Z_6 \approx?\ (y \leftrightarrow x)\,Z_7, y\ \#?\ x\,Z_7\} & (\approx?\text{-app}) \\
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\stackrel{Z_6 := (y \leftrightarrow x)\, Z_7}{\Longrightarrow} & \{y \ \#?\ x\, Z_7\} & (\approx?\text{-susp-2}) \\
\stackrel{\emptyset}{\Longrightarrow} & \{y \ \#?\ x, y \ \#?\ Z_7\} & (\#?\text{-app}) \\
\stackrel{\emptyset}{\Longrightarrow} & \{y \ \#?\ Z_7\} & (\#?\text{-var}) \\
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\stackrel{\epsilon}{\Longrightarrow} & \{y \approx ?\ y,\, Z_6 \approx ?\ (y \leftrightarrow x)\,Z_7,\, y\ \#?\ x\,Z_7\} & (\approx?\text{-app}) \\
\stackrel{\epsilon}{\Longrightarrow} & \{Z_6 \approx ?\ (y \leftrightarrow x)\,Z_7,\, y\ \#?\ x\,Z_7\} & (\approx?\text{-var}) \\
\stackrel{Z_6:=(y\leftrightarrow x)\,Z_7}{\Longrightarrow} & \{y\ \#?\ x\,Z_7\} & (\approx?\text{-susp-2}) \\
\stackrel{\emptyset}{\Longrightarrow} & \{y\ \#?\ x,\, y\ \#?\ Z_7\} & (\#?\text{-app}) \\
\stackrel{\emptyset}{\Longrightarrow} & \{y\ \#?\ Z_7\} & (\#?\text{-var}) \\
\stackrel{y\#Z_7}{\Longrightarrow} & \{\,\} & (\#?\text{-susp})
\end{array}
$$

Solution: $(y \# Z_7, [Z_6 := (y \leftrightarrow x)\, Z_7])$

Conclusion

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Conclusion

- Nominal unification is an algorithm for unifying terms that have names and name-binders.
- Alternatives:
  - Nameless representation (de Bruijn indices [2])
    [but not so intuitive]
  - Higher-order abstract syntax
    [but problem becomes undecidable]
- Current algorithm is exponential. Can use termgraphs and delayed permutations to get polynomial version [1].

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